Example continued: Say X, Y have joint density $f_{X,Y}(x,y) = \begin{cases} 6e^{-2x-3y} & x>0, \\ & y>0 \\ 0 & \text{otherwise} \end{cases}$

Define $Z = \max(X,Y)$. Find the density of Z.

It is often easier to find the CDF first.

$F_Z(a) = P(Z \le a) = P(\max(X,Y) \le a) = P(X \le a \text{ and } Y \le a)$

For $a<0$, $F_Z(a) = 0$, since Z cannot be negative.

For $a>0$, $F_Z(a) = P(X \le a \text{ and } Y \le a) = \displaystyle\int_0^a \boxed{\int_0^a 6e^{-2x-3y}\,dy}\,dx$

$$= \int_0^a \boxed{\frac{6e^{-2x-3y}}{-3}\Bigg|_{y=0}^{a}}\,dx$$

$$= \int_0^a \left(2e^{-2x} - 2e^{-2x-3a}\right)dx$$

$$= \left(\frac{2e^{-2x}}{-2} - \frac{2e^{-2x-3a}}{-2}\right)\Bigg|_{x=0}^{a}$$

$$= \left(-e^{-2a} + e^{-2a-3a}\right) - \left(-1 + e^{-3a}\right)$$

$$= 1 - e^{-3a} - e^{-2a} + e^{-2a-3a}$$

$$= \left(1 - e^{-3a}\right)\left(1 - e^{-2a}\right)$$

One more method!

$F_Z(a) = P(X \le a, Y \le a) = \displaystyle\int_0^a\int_0^a 6e^{-2x-3y}\,dy\,dx = 6\int_0^a e^{-2x}\,dx \int_0^a e^{-3y}\,dy$

$$= 6\left(\frac{e^{-2x}}{-2}\Bigg|_{x=0}^{a}\right)\left(\frac{e^{-3y}}{-3}\Bigg|_{y=0}^{a}\right)$$

$$= \left(1 - e^{-2a}\right)\left(1 - e^{-3a}\right)$$

Still need the density of Z. Know $f_Z(z) = 0$ for $z<0$.

For $z>0$

$f_Z(z) = \dfrac{d}{dz}\left(F_Z(z)\right) = \dfrac{d}{dz}\left[\left(1-e^{-2z}\right)\left(1-e^{-3z}\right)\right]$

$$= \frac{d}{dz}\left(1 - e^{-2z} - e^{-3z} + e^{-5z}\right)$$

$$= 0 - e^{-2z}(-2) - e^{-3z}(-3) + e^{-5z}(-5)$$

$$= 2e^{-2z} + 3e^{-3z} - 5e^{-5z} \quad \text{for } z>0.$$