Density and the CDF for Gamma random variables with parameters λ and r

The density for such a Gamma random variable is

$$f_X(x) = \frac{\lambda^r}{(r-1)!} x^{r-1} e^{-\lambda x},$$

for $x > 0$, and $f_X(x) = 0$ otherwise.

Notice that in the $r = 1$ case, we just have $f_X(x) = \lambda e^{-\lambda x}$ just like it was for Exponential random variables, because a Gamma random variable with $r = 1$ is exactly an Exponential random variable. So the $r = 1$ case has to agree.

What about the CDF of a Gamma random variable? It is

$$F_X(x) = 1 - e^{-\lambda x} \sum_{j=0}^{r-1} \frac{(\lambda x)^j}{j!},$$

for $x > 0$, and $F_X(x) = 0$ otherwise.

Again, let's check the $r = 1$ case. If $r = 1$, then the CDF is $F_X(x) = 1 - e^{-\lambda} x$, and this agrees exactly with the CDF of an Exponential random variable, as it must, because again, a Gamma random variable with parameter $r = 1$ is exactly an Exponential random variable.

Also the expected value of a Gamma random variable must be the sum of r copies the expected value of an Exponential random variable. In other words, if X is a Gamma random variable with parameters λ and r, then X has the same distribution as $X_1 + X_2 + \cdots + X_r$, where the X_j's are independent Exponential random variables. Therefore

$$E(X) = E(X_1 + X_2 + \cdots + X_r) = E(X_1) + E(X_2) + \cdots + E(X_r) = \frac{1}{\lambda} + \frac{1}{\lambda} + \cdots + \frac{1}{\lambda} = \frac{r}{\lambda}.$$

Similarly, since the X_j's are independent random variables (now we need the independence), we can also compute:

$$Var(X) = Var(X_1 + X_2 + \cdots + X_r) = Var(X_1) + Var(X_2) + \cdots + Var(X_r) = \frac{1}{\lambda^2} + \frac{1}{\lambda^2} + \cdots + \frac{1}{\lambda^2} = \frac{r}{\lambda^2}.$$