How do we derive the expected value of a Beta random variable?

Consider the case $\alpha = 3$, $\beta = 8$, so X has density $f_X(x) = 360x^2(1-x)^7$ for $0 < x < 1$

$$E(X) = \int_0^1 (x)(360x^2(1-x)^7)\,dx$$

easier if x has the larger power because we do not have to expand it, and if $(1-x)$ has the smaller power, because then, when we expand it, we get fewer terms.

$u = 1-x \quad du = -dx$

$x = 1-u$

$$= \int_1^0 (1-u)(360)(1-u)^2 u^7 (-1)\,du$$

$$= 360\int_0^1 (1-3u+3u^2-u^3)u^7\,du$$

$$= 360\int_0^1 (u^7 - 3u^8 + 3u^9 - u^{10})\,du$$

$$= 360\left(\frac{u^8}{8} - \frac{3}{9}u^9 + \frac{3}{10}u^{10} - \frac{1}{11}u^{11}\right)\Big|_{u=0}^{1}$$

$$= 360\left(\frac{1}{8} - \frac{3}{9} + \frac{3}{10} - \frac{1}{11}\right)$$

$$= 360\left(\frac{1}{1320}\right) = \frac{3}{11} = \frac{\alpha}{\alpha+\beta}$$ as claimed in the general case.