STAT/MA 41600
Practice Problems: October 15, 2014
Solutions by Mark Daniel Ward

1.

a. We compute $P(3 \le X \le 5) = \int_3^5 \frac{1}{5} e^{-x/5}\,dx = -e^{-x/5}\big|_{x=3}^5 = e^{-3/5} - e^{-1} = 0.1809.$

b. For $a \le 0$, $F_X(a) = 0$ since the density is 0 for $x < a$. For $a > 0$, $F_X(a) = \int_{-\infty}^a f_X(x)\,dx = \int_0^a \frac{1}{5} e^{-x/5}\,dx = -e^{-x/5}\big|_{x=0}^a = 1 - e^{-a/5}$. Thus

$$F_X(x) = \begin{cases} 1 - e^{-x/5} & \text{for } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

c.

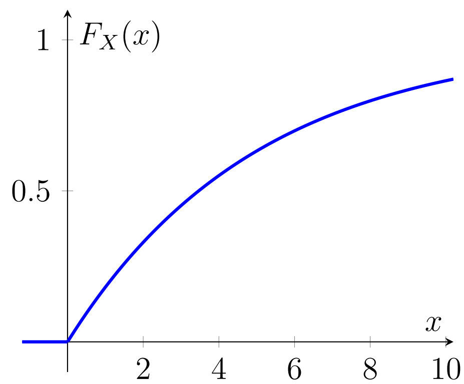

Figure 1: The CDF $F_X(x) = 1 - e^{-x/5}$ of X.

2.

a. We compute

$$1 = \int_0^1 kx^2(1-x)^2\,dx = k\int_0^1 (x^2 - 2x^3 + x^4)\,dx = k\left(\frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5}\right)\Bigg|_{x=0}^1 = k\left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5}\right) = k/30$$

and thus $k = 30$.

b. We compute

$$\int_{3/4}^1 30x^2(1-x)^2\,dx = 30\int_{3/4}^1 (x^2 - 2x^3 + x^4)\,dx = 30\left(\frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5}\right)\Bigg|_{x=3/4}^1 = 53/512 = 0.1035.$$

As an alternative method, we could have used u-substitution with $u = 1 - x$ at the start, so that one of the limits of integration becomes 0. We get

$$\int_0^{1/4} 30(1-u)^2u^2\,du = 30\int_0^{1/4} (u^2 - 2u^3 + u^4)\,dx = 30\left(\frac{u^3}{3} - \frac{2u^4}{4} + \frac{u^5}{5}\right)\Bigg|_{x=0}^{1/4} = 53/512 = 0.1035.$$

3. We have $f_X(x) = k$ for $0 \le x \le 25$, and thus $1 = \int_0^{25} k\,dx = kx|_{x=0}^{25} = 25k$, so $k = 1/25$. Thus $f_X(x) = 1/25$ for $0 \le x \le 25$, and $f_X(x) = 0$ otherwise.

So $P(13.2 \le X \le 19.9) = \int_{13.2}^{19.9} 1/25\,dx = x/25|_{x=13.2}^{19.9} = \frac{19.9-13.2}{25} = 0.268$.

4.

a. Find $P(X > 1/2)$.

We have $P(X > 1/2) = 1 - P(X \le 1/2) = 1 - F_X(1/2) = 1 - (1/2)^4(5 - 4/2) = 1 - (1/16)(6/2) = 1 - 3/16 = 13/16$.

b. The density is the derivative of the CDF. Thus, for $x < 0$ and for $x > 1$, the density is $f_X(x) = 0$. For $0 \le x \le 1$, the density is $f_X(x) = \frac{d}{dx}(x^4(5-4x)) = 4x^3(5-4x) + x^4(-4) = 20x^3 - 20x^4 = 20x^3(1-x)$. So the density of X is

$$f_X(x) = \begin{cases} 20x^3(1-x) & \text{if } 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

5. We use u-substitution with $u = x + 2$, to compute $P(X > 0) = \int_0^1 \frac{\sqrt{3(x+2)}}{6}\,dx = \int_2^3 \frac{\sqrt{3u}}{6}\,du = \frac{\sqrt{3}}{6}\frac{u^{3/2}}{3/2}\Big|_{u=2}^{3} = \frac{\sqrt{3}}{9}\left(3^{3/2} - 2^{3/2}\right) = 1 - \frac{2\sqrt{6}}{9} = 0.4557$.