STAT/MA 41600
Practice Problems: December 10, 2014
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1. a. Since X is uniform and Y has the form $Y = aX + b$ for constants a, b, then Y is uniform too. Notice $2(10) + 2 \le Y \le 2(14) + 2$, i.e., $22 \le Y \le 30$. Also $\frac{1}{30-22} = \frac{1}{8}$. So $f_Y(y) = \frac{1}{8}$ for $22 \le Y \le 30$, and $f_Y(y) = 0$ otherwise.

b. Since the density of Y is constant on $[22, 30]$, then $P(Y > 28) = \frac{\text{length of } [28,30]}{\text{length of } [22,30]} = 2/8 = 1/4$.

c. We have $P(Y > 28) = P(2X + 2 > 28) = P(2X > 26) = P(X > 13)$. Since the density of X is constant on $[10, 14]$, then $P(X > 13) = \frac{\text{length of } [13,14]}{\text{length of } [10,14]} = 1/4$.

2. a. Since X is uniform and Y has the form $Y = aX + b$ for constants a, b, then Y is uniform too. Notice $(1.07)(4) + 3.99 \le Y \le (1.07)(9) + 3.99$, i.e., $8.27 \le Y \le 13.62$. Also $\frac{1}{13.62-8.27} = \frac{1}{5.35}$. So $f_Y(y) = \frac{1}{5.35}$ for $8.27 \le Y \le 13.62$, and $f_Y(y) = 0$ otherwise.

b. *Method #1:* Since Y is uniform on $[8.27, 13.62]$, the expected value of Y is the midpoint of the interval, i.e., $\mathbb{E}(Y) = \frac{8.27+13.62}{2} = 10.945$.

Method #2: We calculate $\mathbb{E}(Y) = \int_{8.27}^{13.62} y \frac{1}{5.35}\, dy = \frac{13.62^2-8.27^2}{2}\frac{1}{5.35} = 10.945$.

c. We calculate $\mathbb{E}(X) = \int_4^9 (1.07x + 3.99)\frac{1}{5}\, dx = \frac{1}{5}(1.07x^2/2 + 3.99x)|_{x=4}^{9} = 10.945$.

3. a. For $8 \le a \le 35$, we have $P(Y \le a) = P((X-1)(X+1) \le a) = P(X^2 - 1 \le a) = P(X^2 \le a+1) = P(X \le \sqrt{a+1}) = \frac{\sqrt{a+1}-3}{6-3}$. Thus, the CDF of Y is

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 8, \\ \frac{\sqrt{y+1}-3}{3} & \text{if } 8 \le y \le 35, \\ 1 & \text{if } 35 < y. \end{cases}$$

b. For $8 \le y \le 35$, we differentiate $F_Y(y)$ with respect to y, and we get $f_Y(y) = \frac{1}{6}(y+1)^{-1/2}$; otherwise, $f_Y(y) = 0$.

c. We use $u = y + 1$ and $du = dy$ to compute $\mathbb{E}(Y) = \int_8^{35} y\, \frac{1}{6}(y+1)^{-1/2}\, dy = \int_9^{36} \frac{1}{6}(u-1)u^{-1/2}\, du = \int_9^{36} \frac{1}{6}(u^{1/2} - u^{-1/2})\, du = \frac{1}{6}(\frac{2}{3}u^{3/2} - 2u^{1/2})|_{u=9}^{36} = \frac{1}{6}(((2/3)(216) - (2)(6)) - ((2/3)(27) - (2)(3))) = 20$.

d. We compute $\mathbb{E}((X-1)(X+1)) = \int_3^6 (x-1)(x+1)(1/3)\, dx = \int_3^6 (1/3)(x^2-1)\, dx = (1/3)(\frac{1}{3}x^3 - x)|_{x=3}^{6} = (1/3)((72-6) - (9-3)) = 20$.

4. We have $\text{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$. Since X, Y have a joint uniform distribution on a triangle with area $(2)(2)/2 = 2$, then $f_{X,Y}(x, y) = 1/2$ on the triangle, and $f_{X,Y}(x, y) = 0$

otherwise. So:

$$\mathbb{E}(XY) = \int_0^2 \int_0^{2-x} xy\frac{1}{2}\,dy\,dx = 1/3,$$

and

$$\mathbb{E}(X) = \int_0^2 \int_0^{2-x} x\frac{1}{2}\,dy\,dx = 2/3,$$

and (since everything is symmetric, we don't even need to calculate):

$$\mathbb{E}(Y) = \int_0^2 \int_0^{2-x} y\frac{1}{2}\,dy\,dx = 2/3.$$

So $\mathrm{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 1/3 - (2/3)(2/3) = -1/9$.

5. Since X_j is Bernoulli with $p = 2/19$, then $\mathrm{Var}(X_j) = (2/19)(17/19) = 34/361$.

Also $\mathrm{Cov}(X_i, X_j) = \mathbb{E}(X_iX_j) - \mathbb{E}(X_i)\mathbb{E}(X_j)$. Also $\mathbb{E}(X_i) = 2/19$ and $\mathbb{E}(X_j) = 2/19$, so we only need $\mathbb{E}(X_iX_j)$. Notice X_iX_j is 0 or 1, i.e., the product X_iX_j is Bernoulli, so $\mathbb{E}(X_iX_j) = P(X_iX_j = 1)$. [We can also see this by $\mathbb{E}(X_iX_j) = 1P(X_iX_j = 1) + 0P(X_iX_j = 0) = P(X_iX_j = 1)$.]

Now we use $P(X_iX_j = 1) = P(X_i = 1 \text{ and } X_j = 1) = P(X_i = 1)P(X_j = 1 \mid X_i = 1)$. We know $P(X_i = 1) = 2/19$. Once $X_i = 1$ is given, there is a row of 18 open seats where the jth couple might sit. The man sits on the end with probability 2/18 and his wife beside him with probability 1/17, or the man does not sit on the end, with probability 16/18 and his wife beside him with probability 2/17, so $P(X_j = 1 \mid X_i = 1) = (2/18)(1/17) + (16/18)(2/17) = 1/9$. [Alternatively, this can be calculated by observing that there are (18)(17) places that they can sit, but there are 17 adjacent pairs of seats, and they can sit in them 2 ways, so $P(X_j = 1 \mid X_i = 1) = \frac{(17)(2)}{(18)(17)} = 2/18 = 1/9$.]

So $\mathbb{E}(X_iX_j) = P(X_iX_j = 1) = (2/19)(1/9)$.

So $\mathrm{Cov}(X_i, X_j) = \mathbb{E}(X_iX_j) - \mathbb{E}(X_i)\mathbb{E}(X_j) = (2/19)(1/9) - (2/19)(2/19) = 2/3249$.

Finally $\mathrm{Var}(X) = \sum_{j=1}^{10} \mathrm{Var}(X_j) + 2\sum\sum_{1\le i<j\le 10} \mathrm{Cov}(X_i, X_j) = (10)(34/361) + (90)(2/3249) = 360/361$.