STAT/MA 41600
In-Class Problem Set #25: October 16, 2015
Solutions by Mark Daniel Ward

1a. We have $P(X \leq 3/4) = \int_0^{3/4} x\,dx = 9/32$.
1b. We have $P(X \leq 5/4) = \int_0^1 x\,dx + \int_1^{5/4}(2-x)\,dx = 1/2 + 7/32 = 23/32$. An easier method is to just calculate $P(X \leq 5/4) = 1 - P(X > 5/4) = 1 - \int_{5/4}^2 (2-x)\,dx = 1 - 9/32 = 23/32$.
1c. For $0 < a < 1$, we have $F_X(a) = \int_0^a x\,dx = a^2/2$. For $1 < a < 2$, we have $F_X(a) = \int_0^1 x\,dx + \int_1^a (2-x)\,dx = 1/2 + (-a^2/2 + 2a - 3/2) = 1 - (2-a)^2/2$; or alternatively $F_X(a) = P(X \leq a) = 1 - P(X > a) = 1 - \int_a^2 (2-x)\,dx = 1 - (a^2/2 - 2a + 2) = 1 - (2-a)^2/2$. So altogether we have

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0, \\ x^2/2 & \text{for } 0 \leq x \leq 1, \\ 1 - (2-x)^2/2 & \text{for } 1 < x \leq 2, \\ 1 & \text{for } x > 2. \end{cases}$$

1d. Yes! We have $F_X(3/4) = (3/4)^2/2 = 9/32$ and $F_X(5/4) = 1 - (2 - 5/4)^2/2 = 23/32$.

2a. For $0 < a < 4$, the triangle where $X + Y \leq a$ has area $a^2/2$, so $P(X + Y \leq a) = (a^2/2)/16 = a^2/32$. Alternatively, we have $\int_0^a \int_0^{a-x} 1/16\,dy\,dx = a^2/32$.
2b. For $4 < a < 8$, the triangle where $X + Y \geq a$ has area $(8-a)^2/2$, so $P(X + Y \geq a) = ((8-a)^2/2)/16 = (8-a)^2/32$. Alternatively, we have $\int_{a-4}^4 \int_{a-x}^4 1/16\,dy\,dx = (8-a)^2/32$. Either way, this yields $P(X + Y \leq a) = 1 - (8-a)^2/32$.
2c. For $0 < w < 4$, we have $F_W(w) = w^2/32$, so $f_W(w) = w/16$. For $4 < w < 8$, we have $F_W(w) = 1 - (8-w)^2/32$, so $f_W(w) = -2(8-w)(-1)/32 = (8-w)/16$. Thus

$$f_W(w) = \begin{cases} w/16 & \text{for } 0 \leq w \leq 4, \\ (8-w)/16 & \text{for } 4 < w \leq 8, \\ 0 & \text{otherwise.} \end{cases}$$

3a. We have $P(Y \geq X) = \int_0^\infty \int_x^\infty 21e^{-3x-7y}\,dy\,dx = \int_0^\infty 3e^{-10x}\,dx = 3/10$.
3b. We have $P(Y \leq 3X) = \int_0^\infty \int_{y/3}^\infty 21e^{-3x-7y}\,dx\,dy = \int_0^\infty 7e^{-8y}\,dy = 7/8$.
3c. We have $P(Y \geq 1/10) = \int_{1/10}^\infty \int_0^\infty 21e^{-3x-7y}\,dx\,dy = \int_{1/10}^\infty 7e^{-7y}\,dy = e^{-7/10}$.

4a. For $x > 0$, we have $f_X(x) = \int_0^\infty 21e^{-3x-7y}\,dy = 3e^{-3x}$, and for $x \leq 0$, we have $f_X(x) = 0$.
4b. For $y > 0$, we have $f_Y(y) = \int_0^\infty 21e^{-3x-7y}\,dx = 7e^{-7y}$, and for $y \leq 0$, we have $f_Y(y) = 0$.
4c. We have $P(Y \geq 1/10) = \int_{1/10}^\infty 7e^{-7y}\,dy = e^{-7/10}$, which agrees with **3c**.