STAT/MA 41600
In-Class Problem Set #44: December 9, 2015
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1a. We have $0.5 \le X \le 2.2$, so $0.3 \le Y \le 0.98$. Since X is uniformly distributed on $[0.5, 2.2]$, and $(2/5)X + 0.1$ is a linear function of X (i.e., just a scaling and shifting), then Y must be uniformly distributed on $[0.3, 0.98]$, so $f_Y(y) = 1/(0.98 - 0.3) = 1/0.68 = 1.47$ for $0.3 \le y \le 0.98$, and $f_Y(y) = 0$ otherwise. If you prefer, we can calculate the CDF of Y. For $0.3 \le y \le 0.98$, we have $P(Y \le y) = \frac{y-0.5}{0.98-0.3}$, and differentiating with respect to y yields $f_Y(y) = 1/(0.98 - 0.3) = 1.47$.

1b. We have $P(Y \le 0.60) = \int_{0.3}^{0.6} 1.47\,dy = (0.3)(1.47) = 0.44$.

1c. We have $P(Y \le 0.60) = P((2/5)X + 0.1 \le 0.60) = P(X \le 1.25) = \frac{1.25-0.5}{2.2-0.5} = 0.44$.

2a. Since Y is uniformly distributed on $[0.3, 0.98]$, then from our formulas for the mean and variance of a Continuous Uniform random variable, we know $\mathbb{E}(Y) = (0.3 + 0.98)/2 = 0.64$ and $\operatorname{Var} Y = (0.98 - 0.3)^2/12 = 0.039$.

We can also calculate: $\mathbb{E}(Y) = \int_{0.3}^{0.98}(y)(1.47)\,dy = 0.64$ and $\mathbb{E}(Y^2) = \int_{0.3}^{0.98}(y^2)(1.47)\,dy = 0.45$ so $\operatorname{Var}(Y) = 0.45 - (0.64)^2 = 0.04$, and the standard deviation is $\sigma_Y = \sqrt{0.04} = 0.2$.

2b. Since Y is a sum of 100 independent random variables, each with mean 0.64 and variance 0.039, then the distribution of Y is approximately Normal with mean $(100)(0.64) = 64$ and variance $(100)(0.039) = 3.9$.

3a. Since $Y = (X+3)(X-3) = X^2 - 9$, and $0 \le X \le 3$, then $-9 \le Y \le 0$. For $-9 \le y \le 0$, we have $F_Y(y) = P(Y \le y) = P(X^2 - 9 \le y) = P(X \le \sqrt{y+9}) = \frac{\sqrt{y+9}-0}{3-0} = \frac{1}{3}\sqrt{y+9}$. Differentiating with respect to y, we get $f_Y(y) = \frac{1}{6}(y+9)^{-1/2}$.

3b. We have $\mathbb{E}(Y) = \int_{-\infty}^{\infty} y\, f_Y(y)\, dy = \int_{-9}^{0}(y)(1/6)(y+9)^{-1/2}\, dy$. Using $u = y + 9$, this gives $\mathbb{E}(Y) = \int_0^9 (u-9)(1/6)(u)^{-1/2}\, du = (1/6)\int_0^9 (u^{1/2} - 9u^{-1/2})\, du = (1/6)((2/3)u^{3/2} - 18u^{1/2})\big|_{u=0}^{9} = (1/6)((2/3)9^{3/2} - (18)9^{1/2}) = (1/6)((2/3)(27) - (18)(3)) = (1/6)(18 - 54) = (1/6)(-36) = -6$.

3c. We compute $\mathbb{E}(Y) = \mathbb{E}((X+3)(X-3)) = \mathbb{E}(X^2 - 9) = \int_0^3 (x^2 - 9)(1/3)\, dx = (1/3)(x^3/3 - 9x)\big|_{x=0}^{3} = (1/3)(3^3/3 - (9)(3)) = (1/3)(9 - 27) = (1/3)(-18) = -6$.

4a. We have $\mathbb{E}(X) = \int_0^5 \int_0^{(2/5)x} (x)(1/5)\, dy\, dx = 10/3$.

4b. We have $\mathbb{E}(Y) = \int_0^5 \int_0^{(2/5)x} (y)(1/5)\, dy\, dx = 2/3$.

4c. We have $\mathbb{E}(XY) = \int_0^5 \int_0^{(2/5)x} (xy)(1/5)\, dy\, dx = 5/2$.

4d. We conclude that $\operatorname{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 5/2 - (10/3)(2/3) = 5/18 = 0.2778$.