STAT/MA 41600
Midterm Exam 2 Answers
Wednesday, November 18, 2015
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1a. We compute $P(Y > X) = \int_0^3 \int_x^4 \frac{1}{36}(3-x)(4-y)\,dy\,dx = \int_0^3 \frac{1}{36}(3-x)(8-4x+x^2/2)\,dx = \int_0^3 \frac{1}{36}(24-20x+(11/2)x^2-x^3/2)\,dx = (1/36)(171/8) = 19/32 = 0.59375$.

1b. Yes, X and Y are independent, because their joint density can be factored, and the joint density is defined in a rectangular region.

1c. We compute $f_X(x) = \int_0^4 \frac{1}{36}(3-x)(4-y)\,dy = \frac{1}{36}(3-x)\int_0^4(4-y)\,dy = \frac{1}{36}(3-x)(8) = (2/9)(3-x)$ for $0 \le x \le 3$, and $f_X(x) = 0$ otherwise.

2a. We have $P(Y > X) = \int_0^\infty \int_x^\infty 10e^{-2x-5y}\,dy\,dx = \int_0^\infty 2e^{-7x}\,dx = 2/7$.

2b. We have $P(Y \le 4X) = \int_0^\infty \int_{y/4}^\infty 10e^{-2x-5y}\,dx\,dy = \int_0^\infty 5e^{-(11/2)y}\,dy = 10/11$.

3a. We have $f_Y(y) = \int_y^\infty 18e^{-2x-7y}\,dx = 9e^{-9y}$ (recall $y > 0$). Thus, we conclude $f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{18e^{-2x-7y}}{9e^{-9y}} = 2e^{-2x+2y}$ for $x > y$, and $f_{X|Y}(x \mid y) = 0$ otherwise.

3b. We can compute $\mathbb{E}(Y) = \int_0^\infty \int_y^\infty (y)(18e^{-2x-7y})\,dx\,dy = \int_0^\infty (y)(9e^{-9y})\,dy = 1/9$, or we could have used the fact that $f_Y(y) = 9e^{-9y}$ for $y > 0$ (from part **3a**), and just observed that Y is Exponential with parameter $\lambda = 9$.

4. We have $\mathbb{E}(\min(X,Y)) = \int_0^3 \int_0^x (y)(1/9)\,dy\,dx + \int_0^3 \int_0^y (x)(1/9)\,dx\,dy = 1/2 + 1/2 = 1$.

5. We compute $P(X < Y) = P(X - Y < 0) = P\big(\frac{X-Y-(175(2)-120(3))}{\sqrt{175(4)+120(9)}} < \frac{0-(175(2)-120(3))}{\sqrt{175(4)+120(9)}}\big) \approx P(Z < 0.24) = 0.5948$.

6a. We have $P(192 < U_1 + \cdots + U_{80} < 208) = P\big(\frac{192-80(5/2)}{\sqrt{80(25/12)}} < \frac{U_1+\cdots+U_{80}-80(5/2)}{\sqrt{80(25/12)}} < \frac{208-80(5/2)}{\sqrt{80(25/12)}}\big) \approx P(-0.62 < Z < 0.62) = P(Z < 0.62) - P(Z \le -0.62) = P(Z < 0.62) - P(Z \ge 0.62) = P(Z < 0.62) - (1 - P(Z < 0.62) = 2P(Z < 0.62) - 1 = 2(0.7324) - 1 = 0.4648$.

6b. We have $P(185 < U_1 + \cdots + U_{80} < 215) = P\big(\frac{185-80(5/2)}{\sqrt{80(25/12)}} < \frac{U_1+\cdots+U_{80}-80(5/2)}{\sqrt{80(25/12)}} < \frac{215-80(5/2)}{\sqrt{80(25/12)}}\big) \approx P(-1.16 < Z < 1.16) = P(Z < 1.16) - P(Z \le -1.16) = P(Z < 1.16) - P(Z \ge 1.16) = P(Z < 1.16) - (1 - P(Z < 1.16) = 2P(Z < 1.16) - 1 = 2(0.8770) - 1 = 0.7540$.

Question 1 was like question #5 on the Practice Problem Set 25, with some small changes.
Question 2 was like question #3ab on the 2015 Problem Set 25, with some small changes.
Question 3 was like question #3a/4a on the 2014 Problem Set 27, with small changes.
Question 4 was like question #2 on the 2015 Problem Set 31, with some small changes.
Question 5 was like question #2 on the 2014 Problem Set 37, with some small changes.
Question 6 was like question #3 on the 2014 Problem Set 37, with some small changes.