STAT/MA 41600
In-Class Problem Set #12: September 19, 2016
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Problem Set 12 Answers

1a. We have $\mathbb{E}(X^2) = (3^2)(2/15) + (2^2)(1/5) + (1^2)(2/5) + (0^2)(4/15) = 12/5$.
1b. We have $\mathbb{E}(X^2) = \mathbb{E}((X_1 + X_2 + X_3)^2)$, which has 6 terms of the form $\mathbb{E}(X_iX_j)$ (for $i \neq j$) and 3 terms of the form $\mathbb{E}(X_j^2)$. We have $\mathbb{E}(X_iX_j) = (2/5)(1/2) = 1/5$. Also, since indicators only take on values 0 or 1, then $\mathbb{E}(X_j^2) = \mathbb{E}(X_j) = 2/5$. Thus $\mathbb{E}(X^2) = (6)(1/5) + (3)(2/5) = 12/5$.
1c. We have $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 12/5 - (6/5)^2 = 24/25$.

2a. We have $\mathbb{E}(X^2) = (2^2)(105/221) + (1^2)(96/221) + (0^2)(20/221) = 516/221$.
2b. We have $\mathbb{E}(X^2) = \mathbb{E}((X_1 + X_2)^2)$, which has 2 terms of the form $\mathbb{E}(X_iX_j)$ (for $i \neq j$) and 2 terms of the form $\mathbb{E}(X_j^2)$. We have $\mathbb{E}(X_iX_j) = (36/52)(35/51) = 105/221$. Also, since indicators only take on values 0 or 1, then $\mathbb{E}(X_j^2) = \mathbb{E}(X_j) = 36/52$. Thus $\mathbb{E}(X^2) = (2)(105/221) + (2)(36/52) = 516/221$.

Or, with the formulation from the previous problem set with $X = X_1 + \cdots + X_{36}$, we have $\mathbb{E}(X^2) = \mathbb{E}((X_1 + \cdots + X_{36})^2)$, which has $36^2 - 36 = 1296 - 36 = 1260$ terms of the form $\mathbb{E}(X_iX_j)$ (for $i \neq j$) and 36 terms of the form $\mathbb{E}(X_j^2)$. We have $\mathbb{E}(X_iX_j) = (2/52)(1/51) = 1/1326$. Also, since indicators only take on values 0 or 1, then $\mathbb{E}(X_j^2) = \mathbb{E}(X_j) = 2/52 = 1/26$. Thus $\mathbb{E}(X^2) = (1260)(1/1326) + (36)(1/26) = 516/221$.
2c. We have $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 516/221 - (18/13)^2 = 1200/2873$.

3. We have $\mathbb{E}(X^2) = \mathbb{E}((X_1 + X_2 + X_3)^2)$, which has 6 terms of the form $\mathbb{E}(X_iX_j)$ (for $i \neq j$) and 3 terms of the form $\mathbb{E}(X_j^2)$. We have $\mathbb{E}(X_iX_j) = (36/52)(35/51) = 105/221$. Also, since indicators only take on values 0 or 1, then $\mathbb{E}(X_j^2) = \mathbb{E}(X_j) = 36/52$. Thus $\mathbb{E}(X^2) = (6)(105/221) + (3)(36/52) = 1089/221$.
3c. We have $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 1089/221 - (27/13)^2 = 1764/2873$.

4a. We have $\mathbb{E}(X^2) = (6^2)(1/6) + (5^2)(1/6) + (4^2)(7/24) + (3^2)(5/24) + (2^2)(3/24) + (1^1)(1/24) = 69/4$.
4b. We have $\mathbb{E}(X^2) = \mathbb{E}((X_1 + \cdots + X_6)^2)$, which has:
10 terms of the form $\mathbb{E}(X_iX_6) = \mathbb{E}(X_6)$ for $i < 6$;
8 terms of the form $\mathbb{E}(X_iX_5) = \mathbb{E}(X_5)$ for $i < 5$;
6 terms of the form $\mathbb{E}(X_iX_4) = \mathbb{E}(X_4)$ for $i < 4$;
4 terms of the form $\mathbb{E}(X_iX_3) = \mathbb{E}(X_3)$ for $i < 3$;
2 terms of the form $\mathbb{E}(X_iX_2) = \mathbb{E}(X_2)$ for $i < 2$;
and of course the terms of the form $\mathbb{E}(X_j^2) = \mathbb{E}(X_j)$.
So we get $\mathbb{E}(X^2) = (11)(1/6) + (9)(1/3) + (7)(5/8) + (5)(5/6) + (3)(23/24) + (1)(1) = 69/4$.
4c. We have $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 69/4 - (47/12)^2 = 275/144$.