STAT/MA 41600
In-Class Problem Set #27: October 19, 2016
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**Problem Set 27 Answers**

**1a.** Yes, it is true, since $X$ and $Y$ are independent.
**1b.** Since $X$ and $Y$ are independent, we have $P(Y > 3/10 \mid X > 1/10) = P(Y > 3/10)$. As observed in 1a of the solutions to the previous problem set, we have $f_Y(y) = 3e^{-3y}$ for $y > 0$, and $f_Y(y) = 0$ otherwise. Thus $P(Y > 3/10) = \int_{3/10}^{\infty} 3e^{-3y}\,dy = -e^{-3y}|_{y=3/10}^{\infty} = e^{-9/10}$.

**2a.** We have $P(Y > 3/10 \mid X > 1/10) = \frac{P(Y>3/10\ \&\ X>1/10)}{P(X>1/10)} = \frac{P(Y>3/10\ \&\ X>1/10)}{e^{-8/10}}$. We compute $P(Y > 3/10\,\&\,X > 1/10) = \int_{3/10}^{\infty}\int_{1/10}^{y} 24e^{-5x-3y}\,dx\,dy = \int_{3/10}^{\infty} -(24/5)e^{-5x-3y}|_{x=1/10}^{y}\,dy = \int_{3/10}^{\infty}((24/5)e^{-1/2-3y} - (24/5)e^{-8y})\,dy = (-(8/5)e^{-1/2-3y} + (3/5)e^{-8y})|_{y=3/10}^{\infty} = (8/5)e^{-1/2-3(3/10)} - (3/5)e^{-8(3/10)} = (8/5)e^{-14/10} - (3/5)e^{-24/10}$. So altogether we get $P(Y > 3/10 \mid X > 1/10) = \frac{(8/5)e^{-14/10}-(3/5)e^{-24/10}}{e^{-8/10}} = (8/5)e^{-6/10} - (3/5)e^{-16/10}$.
**2b.** We have $f_X(x) = \int_x^{\infty} 24e^{-5x-3y}\,dy = -8e^{-5x-3y}|_{y=x}^{\infty} = 8e^{-8x}$ for $x > 0$, and $f_X(x) = 0$ otherwise, so $f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{24e^{-5x-3y}}{8e^{-8x}} = 3e^{3x-3y}$ for $y > x$, and $f_{Y|X}(y \mid x) = 0$ otherwise.
**2c.** We have $P(Y > 3/10 \mid X = 1/10) = \int_{3/10}^{\infty} f_{Y|X}(y \mid 1/10)\,dy \int_{3/10}^{\infty} 3e^{3/10-3y}\,dy - e^{3/10-3y}|_{y=3/10}^{\infty} = e^{-6/10}$
**3a.** Since the joint pdf of $X$ and $Y$ is constant on a triangle with area 16, then $f_{X,Y}(x, y) = 1/16$ for $X, Y$ in the triangle, and $f_{X,Y}(x, y) = 0$ otherwise. Also $f_X(x) = \int_0^{4-x/2} 1/16\,dy = (1/16)(4 - x/2)$. Thus, we have $f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{1/16}{(1/16)(4-x/2)} = 1/(4 - x/2)$, and in particular, $f_{Y|X}(y \mid x) = 1/3$.
**3b.** We have $P(Y > 1 \mid X = 2) = \int_1^3 f_{Y|X}(y \mid 2)\,dy = \int_1^3 1/3\,dy = 2/3$.
**3c.** We have $P(Y > 1 \mid X > 2) = \frac{P(Y>1\ \&\ X>2)}{P(X>2)} = \frac{\int_1^3\int_2^6 1/16\ dx\ dy}{\int_0^3\int_2^8 1/16\ dx\ dy} = \frac{(1/16)(4)(2)/2}{(1/16)(6)(3)/2} = 4/9$.

**4a.** We have $f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{(1/12)(4-xy)}{2/3-x/6}$ for $0 < x < 2$ and $0 < y < 2$.
**4b.** We have $f_{Y|X}(y \mid 2/3) = \frac{f_{X,Y}(2/3,y)}{f_X(2/3)} = \frac{(1/12)(4-(2/3)y)}{2/3-(2/3)/6} = (3/20)(4 - (2/3)y)$ for $0 < y < 2$, and $f_{Y|X}(y \mid 2/3) = 0$ otherwise. Therefore, we compute $P(Y < 4/3 \mid X = 2/3) = \int_0^{4/3}(3/20)(4 - (2/3)y)\,dy = (3/20)(4y - (2/3)y^2/2)|_{y=0}^{4/3} = 32/45$
**4c.** We compute $P(Y < 4/3 \mid X < 2/3) = \frac{P(Y<4/3\ \&\ X<2/3)}{P(X<2/3)} = \frac{\int_0^{4/3}\int_0^{2/3}(1/12)(4-xy)\ dx\ dy}{\int_0^2\int_0^{2/3}(1/12)(4-xy)\ dx\ dy} = \frac{\int_0^{4/3}(1/12)(4x-yx^2/2)|_{x=0}^{2/3}\ dy}{\int_0^2(1/12)(4x-yx^2/2)|_{x=0}^{2/3}\ dy} = \frac{\int_0^{4/3}(1/12)(8/3-4y/18)\ dy}{\int_0^2(1/12)(8/3-4y/18)\ dy} = \frac{(1/12)((8/3)y-4y^2/36)|_{y=0}^{4/3}}{(1/12)((8/3)y-4y^2/36)|_{y=0}^{2}} = \frac{(1/12)((8/3)(4/3)-4(4/3)^2/36)}{(1/12)((8/3)(2)-4(2)^2/36)} = 68/99$.