STAT/MA 41600
In-Class Problem Set #44: December 7, 2016
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Problem Set 44 Answers

1a. We compute $P(X > 5) = P(-\ln(U/3) > 5) = P(\ln(U/3) < -5) = P(U/3 < e^{-5}) = P(U < 3e^{-5}) = \frac{3e^{-5}-0}{3-0} = e^{-5}$.
1b. Using a instead of 5, the same argument shows that $P(X > a) = e^{-a}$.
1c. Since $P(X > a) = e^{-a}$ for $a > 0$, then X is an exponential random variable with $\lambda = 1$.

2a. We see that Y is nonnegative. For $a \geq 0$, we have $P(Y \geq a) = P(X^2 \geq a) = P(X \geq \sqrt{a}) = e^{-4\sqrt{a}}$, where the second equality is true since X is nonnegative, and the third equality is true since X is exponential with $\lambda = 4$. Thus, if $y \geq 0$, we have $F_Y(y) = 1 - P(Y \geq y) = 1 - e^{-4\sqrt{y}}$, so $f_Y(y) = -e^{-4\sqrt{y}}(-4)(1/2)y^{-1/2} = 2y^{-1/2}e^{-4\sqrt{y}}$.
2b. We get $\mathbb{E}(Y) = \int_0^\infty (y)(2y^{-1/2}e^{-4\sqrt{y}})\,dy = \int_0^\infty 2\sqrt{y}\,e^{-4\sqrt{y}}\,dy$. Then we use $u = \sqrt{y}$ and $du = (1/2)y^{-1/2}\,dy$, so $\mathbb{E}(Y) = \int_0^\infty 4u^2e^{-4u}\,du$, which equals 1/8, using integration by parts two times.
2c. We have $\mathbb{E}(X) = 1/\lambda = 1/4$ and $\mathrm{Var}(Y) = 1/\lambda^2 = 1/16$, so $\mathbb{E}(X^2) = \mathrm{Var}(X) + (\mathbb{E}(X))^2 = 1/16 + (1/4)^2 = 1/8$, which agrees with $\mathbb{E}(Y)$.

3a. For $0 < a < 8$, we have $P(X < a) = P(U^3 < a) = P(U < a^{1/3}) = \frac{a^{1/3}-0}{2-0} = a^{1/3}/2$, so $f_X(x) = (1/2)(1/3)x^{-2/3} = x^{-2/3}/6$.
3b. We have $\mathbb{E}(X) = \int_0^8 (x)(x^{-2/3}/6)\,dx = \int_0^8 x^{1/3}/6\,dx = x^{4/3}/8|_{x=0}^8 = 8^{1/3} = 2$.
3c. We have $\mathbb{E}(U^3) = \int_0^2 (u^3)(1/2)\,du = u^4/8|_{u=0}^2 = 16/8 = 2$.

4a. We have $\mathbb{E}(X) = \int_0^2\int_0^{4-x}(x)(1/6)\,dy\,dx = \int_0^2 (x)(1/6)(y)|_{y=0}^{4-x}\,dx = \int_0^2 (x)(1/6)(4-x)\,dx = \int_0^2 (1/6)(4x - x^2)\,dx = (1/6)(2x^2 - x^3/3)|_{x=0}^2 = (1/6)(8 - 8/3) = 8/9$.
4b. We have $\mathbb{E}(Y) = \int_0^2\int_0^{4-x}(y)(1/6)\,dy\,dx = \int_0^2 y^2/12|_{y=0}^{4-x}\,dx = \int_0^2 (4-x)^2/12\,dx = \int_0^2 (16 - 8x + x^2)/12\,dx = (16x - 4x^2 + x^3/3)/12|_{x=0}^2 = (32 - 16 + 8/3)/12 = 14/9$.
4c. We have $\mathbb{E}(XY) = \int_0^2\int_0^{4-x}(xy)(1/6)\,dy\,dx = \int_0^2 xy^2/12|_{y=0}^{4-x}\,dx = \int_0^2 x(4-x)^2/12\,dx = \int_0^2 (16x - 8x^2 + x^3)/12\,dx = (8x^2 - 8x^3/3 + x^4/4)/12|_{x=0}^2 = (32 - 64/3 + 4)/12 = 11/9$.
4d. We conclude that $\mathrm{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 11/9 - (8/9)(14/9) = -13/81$.