STAT/MA 41600
In-Class Problem Set #12: September 18, 2017
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Problem Set 12 Answers

1a. We have $\mathbb{E}(X^2) = (2^2)(1/24) + (3^2)(2/24) + (4^2)(3/24) + (5^2)(4/24) + (6^2)(4/24) + (7^2)(4/24) + (8^2)(3/24) + (9^2)(2/24) + (10^2)(1/24) = 241/6$.
1b. We get $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 241/6 - 6^2 = 25/6$.
1c. We have $\mathbb{E}(X_1^2) = (1/4)(1^2 + 2^2 + 3^2 + 4^2) = 15/2$, so $\text{Var}(X_1) = 15/2 - (5/2)^2 = 5/4$. We have $\mathbb{E}(X_2^2) = (1/6)(1^2 + \cdots + 6^2) = 91/6$, so $\text{Var}(X_2) = 91/6 - (7/2)^2 = 35/12$. So we conclude $\text{Var}(X) = \text{Var}(X_1) + \text{Var}(X_2) = 5/4 + 35/12 = 25/6$.

2a. We have $\mathbb{E}(X^2) = (0^2)(1/32) + (1^2)(5/32) + (2^2)(10/32) + (3^2)(10/32) + (4^2)(5/32) + (5^2)(1/32) = 15/2$.
2b. We compute $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 15/2 - (5/2)^2 = 5/4$.
2c. Since each X_j is an indicator random variable, then $\mathbb{E}(X_j^2)$ and $\mathbb{E}(X_j)$ are the same; in this case, they are each 1/2. So we have $\text{Var}(X_j) = \mathbb{E}(X_j^2) - (\mathbb{E}(X_j))^2 = 1/2 - (1/2)^2 = 1/4$. We conclude that $\text{Var}(X) = \text{Var}(X_1) + \cdots + \text{Var}(X_5) = 1/4 + \cdots + 1/4 = 5/4$.

3a. We have $\mathbb{E}(X^2) = (0^2)\binom{4}{0}\binom{48}{5}/\binom{52}{5} + (1^2)\binom{4}{1}\binom{48}{4}/\binom{52}{5} + (2^2)\binom{4}{2}\binom{48}{3}/\binom{52}{5} + (3^2)\binom{4}{3}\binom{48}{2}/\binom{52}{5} + (4^2)\binom{4}{4}\binom{48}{1}/\binom{52}{5} + (5^2\binom{4}{5}\binom{48}{0}/\binom{52}{5} = (0^2)(35673/54145) + (1^2)(3243/10829) + (2^2)(2162/54145) + (3^2)(94/54145) + (4^2)(1/54145) + (5^2)(0) = 105/221$.
3b. We have $\mathbb{E}(X^2) = \mathbb{E}((X_1 + \cdots + X_5)(X_1 + \cdots + X_5)) = 5\mathbb{E}(X_1^2) + 20\mathbb{E}(X_1X_2)$. We have $\mathbb{E}(X_1^2) = \mathbb{E}(X_1) = 4/52 = 1/13$ and $\mathbb{E}(X_1X_2) = P(X_1X_2 = 1) = P(X_1 = 1)P(X_2 = 1 \mid X_1 = 1) = (4/52)(3/51) = 1/221$. So we conclude that $\mathbb{E}(X^2) = (5)(1/13) + (20)(1/221) = 105/221$.
3c. We compute $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 105/221 - (5/13)^2 = 940/2873$.
3d. We have $\mathbb{E}(X^2) = (0^2)\binom{5}{0}(4/52)^0(48/52)^5 + (1^2)\binom{5}{1}(4/52)^1(48/52)^4 + (2^2)\binom{5}{2}(4/52)^2(48/52)^3 + (3^2)\binom{5}{3}(4/52)^3(48/52)^2 + (4^2)\binom{5}{4}(4/52)^4(48/52)^1 + (5^2\binom{5}{5}(4/52)^5(48/52)^0 = (0^2)(248832/371293) + (1^2)(103680/371293) + (2^2)(17280/371293) + (3^2)(1440/371293) + (4^2)(60/371293) + (5^2)(1/371293) = 85/169$.
3e. We compute $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 85/169 - (5/13)^2 = 60/169$.
3f. Since each X_j is an indicator random variable, then $\mathbb{E}(X_j^2)$ and $\mathbb{E}(X_j)$ are the same; in this case, they are each 1/13. So we have $\text{Var}(X_j) = \mathbb{E}(X_j^2) - (\mathbb{E}(X_j))^2 = 1/13 - (1/13)^2 = 12/169$. We conclude that $\text{Var}(X) = \text{Var}(X_1) + \cdots + \text{Var}(X_5) = 12/169 + \cdots + 12/169 = 60/169$.

4a. We have $\mathbb{E}(X^2) = (0^2)(1/20) + (1^2)(9/20) + (2^2)(9/20) + (3^2)(1/20) = 27/10$.
4b. We have $\mathbb{E}(X^2) = \mathbb{E}((X_1 + X_2 + X_3)(X_1 + X_2 + X_3)) = 3\mathbb{E}(X_1^2) + 6\mathbb{E}(X_1X_2)$. We have $\mathbb{E}(X_1^2) = \mathbb{E}(X_1) = 3/6 = 1/2$ and $\mathbb{E}(X_1X_2) = P(X_1X_2 = 1) = P(X_1 = 1)P(X_2 = 1 \mid X_1 = 1) = (3/6)(2/5) = 1/5$. So we conclude that $\mathbb{E}(X^2) = (3)(1/2) + (6)(1/5) = 27/10$.
4c. We compute $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 27/10 - (3/2)^2 = 9/20$.