STAT/MA 41600
In-Class Problem Set #16: September 22, 2017
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Problem Set 16 Answers

1ab. The probability of having a triple on a round is $p = 1/6^2 = 1/36$. So the expected number of rounds is $1/p = 36$ and the variance is $(1-p)/p^2 = 1260$.
1c. The expected winnings on a round are $(1/36)(100) + (35/36)(0) = 100/36 = 25/9 = 2.78$. Therefore, the expected loses on a round must be $-25/9 = -2.78$, if the game is to be "fair". So she should charge \$2.78 per ticket.

2a. The probability of no chocolate chip cookie on Monday through Friday is $(0.60)^5 = 0.07776$.
2b. The probability of no chocolate chip cookie on Monday through Wednesday is $(0.60)^3 = 0.216$.
2c. We use x to represent the number of days without a chocolate cookie. Then the probability that he finally gets his first chocolate chip cookie on a Monday is $\sum_{x=0}^{\infty}(.40)(.60)^{7x} = \frac{0.40}{1-0.60^7} = 0.4115$.

3a. On a given day, the probability that none of them get a chocolate cookie is $(0.60)^3$. So X is a Geometric random variable with $p = 1-(0.60)^3$. Therefore, we get $P(X = 5) = ((0.60)^3)^4(1-(0.60)^3) = 0.001707$.
3b. We have $\mathbb{E}(X) = 1/(1-(0.60)^3) = 1.2755$.

4a. We compute:

$$\begin{aligned}
P(X > Y) &= \sum_{y=1}^{\infty}\sum_{x=y+1}^{\infty}(1-p)^{x-1}p(1-r)^{y-1}r \\
&= \sum_{y=1}^{\infty}p(1-r)^{y-1}r\sum_{x=y}^{\infty}(1-p)^x \\
&= \sum_{y=1}^{\infty}p(1-r)^{y-1}r(1-p)^y/(1-(1-p)) \\
&= r(1-p)\sum_{y=1}^{\infty}(1-r)^{y-1}(1-p)^{y-1} \\
&= r(1-p)\sum_{y=0}^{\infty}((1-r)(1-p))^y \\
&= r(1-p)/(1-(1-r)(1-p))
\end{aligned}$$

4b. By symmetry, we have $P(Y > X) = p(1-r)/(1-(1-r)(1-p))$.
4c. The probability X and Y are equal is $P(X = Y) = \sum_{n=1}^{\infty}P(X = Y = n) = \sum_{n=1}^{\infty}(1-p)^{n-1}p(1-r)^{n-1}r = pr\sum_{n=1}^{\infty}((1-p)^{n-1}(1-r)^{n-1}) = pr/(1-(1-p)(1-r))$.

We can check that these three answers sum to 1:

$$\begin{aligned}
\frac{r(1-p)}{1-(1-r)(1-p)} + \frac{p(1-r)}{1-(1-r)(1-p)} + \frac{pr}{1-(1-p)(1-r)} &= \frac{r-rp+p-pr+pr}{1-(1-p)(1-r)} \\
&= \frac{r-rp+p}{1-(1-p)(1-r)} \\
&= 1
\end{aligned}$$