STAT/MA 41600
In-Class Problem Set #19: September 29, 2017
Solutions by Mark Daniel Ward

Problem Set 19 Answers

1. The probability X is even is $P(X=0)+P(X=2) = \frac{\binom{4}{0}\binom{6}{3}}{\binom{10}{3}} + \frac{\binom{4}{2}\binom{6}{1}}{\binom{10}{3}} = 1/6+3/10 = 7/15$.
The probability X is odd is $P(X=1)+P(X=3) = \frac{\binom{4}{1}\binom{6}{2}}{\binom{10}{3}} + \frac{\binom{4}{3}\binom{6}{0}}{\binom{10}{3}} = 1/2+1/30 = 8/15$.
So X is slightly more likely to be odd.

2a. We see that X is a Binomial random variable with parameters $n=3$ and $p=1/6$.
2b. We see that Y is a Hypergeometric random variable with parameters $N=52$, $M=4$, and $n=5$.
2c. We compute $P(X \geq Y) = P(X=3 \;\&\; Y=0) + P(X=3 \;\&\; Y=1) + P(X=3 \;\&\; Y=2) + P(X=3 \;\&\; Y=3) + P(X=2 \;\&\; Y=0) + P(X=3 \;\&\; Y=1) + P(X=3 \;\&\; Y=2) + P(X=1 \;\&\; Y=0) + P(X=3 \;\&\; Y=1) + P(X=0 \;\&\; Y=0)$,

By factoring, we have $P(X \geq Y) = P(X=3)(P(Y=0)+P(Y=1)+P(Y=2)+P(Y=3)) + P(X=2)(P(Y=0)+P(Y=1)+P(Y=2)) + P(X=1)(P(Y=0)+P(Y=1)) + P(X=0 \;\&\; Y=0)$.

So we get

$$\begin{aligned}
P(X \geq Y) = &\binom{3}{3}(1/6)^3(5/6)^0\left(\frac{\binom{4}{0}\binom{48}{5}}{\binom{52}{5}} + \frac{\binom{4}{1}\binom{48}{4}}{\binom{52}{5}} + \frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}} + \frac{\binom{4}{3}\binom{48}{2}}{\binom{52}{5}}\right)\\
&+ \binom{3}{2}(1/6)^2(5/6)^1\left(\frac{\binom{4}{0}\binom{48}{5}}{\binom{52}{5}} + \frac{\binom{4}{1}\binom{48}{4}}{\binom{52}{5}} + \frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}}\right)\\
&+ \binom{3}{1}(1/6)^1(5/6)^2\left(\frac{\binom{4}{0}\binom{48}{5}}{\binom{52}{5}} + \frac{\binom{4}{1}\binom{48}{4}}{\binom{52}{5}}\right)\\
&+ \binom{3}{0}(1/6)^0(5/6)^3\left(\frac{\binom{4}{0}\binom{48}{5}}{\binom{52}{5}}\right)\\
= &\frac{752}{162435} + \frac{27025}{389844} + \frac{10810}{32487} + \frac{297275}{779688} = \frac{438839}{556920}\\
= &\,0.00463 + 0.06932 + 0.33275 + 0.38127 = 0.78797
\end{aligned}$$

3a. We see that X is a Hypergeometric with parameters $N=1000$, $M=7$, and $n=10$.
3b. We have $P(X=1) = \binom{7}{1}\binom{993}{9}/\binom{1000}{10} = 0.06629$.
3c. We see that Y is a Binomial random variable with parameters $n=10$ and $p=7/1000$.
3d. We have $P(Y=1) = \binom{10}{1}(7/1000)^1(993/1000)^9 = 0.06571$.
3e. Yes, these are pretty close, because a Binomial random variable can often be used to approximate a Hypergeometric random variable.

4ab. Since X is a Hypergeometric random variable with parameters $N=30$, $M=20$, and $n=7$, then $\mathbb{E}(X) = nM/N = 14/3$ and $\text{Var}(X) = nM/N(1-M/N)(N-n)/(N-1) = 322/261$.