STAT/MA 41600
In-Class Problem Set #26: October 16, 2017
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Problem Set 26 Answers

1a. When $y < x$, then $x - y$ is positive, and thus $f_{X,Y}(x,y)$ is positive too. When $y \geq x$, then $f_{X,Y}(x,y) = 0$. So $f_{X,Y}(x,y)$ is always nonnegative. We also check that $\int_0^2 \int_0^x (3/4)(x-y)\,dy\,dx = \int_0^2 (3/4)(xy - y^2/2)|_{y=0}^{x}\,dx = \int_0^2 (3/4)(x^2/2)\,dx = (3/4)(x^3/6)|_{x=0}^{2} = (3/4)(8/6) = 1$.
1b. For $0 < x < 2$, the probability density function of X is $f_X(x) = \int_0^x (3/4)(x-y)\,dy = (3/4)(xy - y^2/2)|_{y=0}^{x} = (3/4)(x^2/2) = 3x^2/8$, and $f_X(x) = 0$ otherwise.
1c. For $0 < y < 2$, the probability density function of Y is $f_Y(y) = \int_y^2 (3/4)(x-y)\,dx = (3/4)(x^2/2 - yx)|_{x=y}^{2} = (3/4)((2-2y) - (y^2/2 - y^2)) = (3/4)(2 - 2y + y^2/2)$, and $f_Y(y) = 0$ otherwise.

2a. The random variables X and Y are not independent, because we cannot factor $f_{X,Y}(x,y)$ into a function of x times a function of y.
2b. We compute $P(X+Y \leq 1) = \int_0^{1/2}\int_y^{1-y}(3/4)(x-y)\,dx\,dy = \int_0^{1/2}(3/4)(x^2/2 - yx)|_{x=y}^{1-y}\,dy = \int_0^{1/2}(3/4)(((1-y)^2/2 - y(1-y)) - (y^2/2 - y^2))\,dy = \int_0^{1/2}(3/4)(1/2 - 2y + 2y^2)\,dy = (3/4)(y/2 - y^2 + 2y^3/3)|_{y=0}^{1/2} = (3/4)(1/4 - 1/4 + 1/12) = 1/16$.

3. We have $P(X > 2Y) = \int_0^\infty \int_{2y}^\infty (3e^{-3x})(8e^{-8y})\,dx\,dy = \int_0^\infty (-e^{-3x})(8e^{-8y})|_{x=2y}^{\infty}\,dy = \int_0^\infty (e^{-6y})(8e^{-8y})\,dy = \int_0^\infty 8e^{-14y}\,dy = -(8/14)e^{-14y}|_{y=0}^{\infty} = 4/7$.

4a. The joint probability density function is $f_{X,Y}(x,y) = 1/(9\pi)$ for points (x,y) in the circle with center at the origin and radius 3, and $f_{X,Y}(x,y) = 0$ otherwise.
4b. The probability that $X^2 + Y^2$ is less than 4 is $4\pi/(9\pi)$, so the probability that $X^2 + Y^2$ is larger than 4 is $1 - 4\pi/(9\pi) = 5/9$.