STAT/MA 41600
In-Class Problem Set #27: October 18, 2017
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Problem Set 27 Answers

1a. We have $f_X(1) = \int_0^1 (3/4)(1-y)\,dy = (3/4)(y - y^2/2)|_{y=0}^1 = (3/4)(1-1/2) = 3/8$. Therefore, we get $f_{Y|X}(y \mid 1) = \frac{f_{X,Y}(1,y)}{f_X(1)} = \frac{(3/4)(1-y)}{3/8} = 2(1-y)$. So $P(Y < 1/2 \mid X = 1) = \int_0^{1/2} 2(1-y)\,dy = 2(y - y^2/2)|_{y=0}^{1/2} = 2(1/2 - (1/2)^2/2) = 3/4$.

1b. We have $P(Y < 1/2 \mid X < 1) = \frac{P(Y<1/2 \ \&\ X<1)}{P(X<1)} = \frac{\int_0^{1/2}\int_y^1 (3/4)(x-y)\,dx\,dy}{\int_0^1\int_y^1 (3/4)(x-y)\,dx\,dy} = \frac{\int_0^{1/2}(3/4)(x^2/2-yx)|_{x=y}^1\,dy}{\int_0^1 (3/4)(x^2/2-yx)|_{x=y}^1\,dy} = \frac{\int_0^{1/2}(3/4)((1^2/2-y)-(y^2/2-y^2))\,dy}{\int_0^1(3/4)((1^2/2-y)-(y^2/2-y^2))\,dy} = \frac{\int_0^{1/2}(3/4)(1/2-y+y^2/2)\,dy}{\int_0^1(3/4)(1/2-y+y^2/2)\,dy} = \frac{(3/4)(y/2-y^2/2+y^3/6)|_{y=0}^{1/2}}{(3/4)(y/2-y^2/2+y^3/6)|_{y=0}^1} = \frac{7/64}{1/8} = 7/8$.

2a. For $y \le 0$, we have $f_Y(y) = 0$. For $y > 0$, we get $f_Y(y) = \int_{5y}^\infty 69e^{-3x-8y}\,dx = -(69/3)e^{-3x-8y}|_{y=5x}^\infty = (69/3)e^{-3(5y)-8y} = 23e^{-23y}$.

2b. We compute $P(Y > 1/20) = \int_{1/20}^\infty 23e^{-23y}\,dy = -e^{-23y}|_{y=1/20}^\infty = e^{-23/20} = 0.3166$.

3a. We have $f_{X|Y}(x \mid 1/15) = \frac{f_{X,Y}(x,1/15)}{f_Y(1/15)} = \frac{69e^{-3x-8/15}}{23e^{-23/15}} = (69/23)e^{-3x-8/15+23/15} = (69/23)e^{-3x+1}$ for $x > (5)(1/15) = 1/3$, and $f_{X|Y}(x \mid 1/15) = 0$ otherwise.

3b. We have $P(X > 1/2 \mid Y = 1/15) = \int_{1/2}^\infty f_{X|Y}(x \mid 1/15)\,dx = \int_{1/2}^\infty (69/23)e^{-3x+1}\,dx = -e^{-3x+1}|_{x=1/2}^\infty = e^{-3/2+1} = e^{-1/2} = 0.6065$.

4. The joint probability density function is $f_{X,Y}(x,y) = 1/12$ for (x,y) in the triangle, and $f_{X,Y}(x,y) = 0$ otherwise. We also have $f_Y(1) = \int_{-2}^4 1/12\,dx = 1/2$. Therefore, we get $f_{X|Y}(x \mid 1) = \frac{f_{X,Y}(x,1)}{f_Y(1)} = \frac{1/12}{1/2} = 1/6$ for $-2 < x < 4$, and $f_{X|Y}(x \mid 1) = 0$ otherwise. Therefore, we get $P(X > 0 \mid Y = 1) = \int_0^4 f_{X|Y}(x \mid y)\,dx = \int_0^4 1/6\,dx = 4/6 = 2/3$.