STAT/MA 41600
In-Class Problem Set #32: October 27, 2017
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Problem Set 32 Answers

1a. Let X be the number of minutes after 12 noon when the taxi cab arrives. Then $P(X > 5) = e^{-5/3} = .1889$.
1b. We have $P(X > 7 \mid X > 5) = \frac{P(X>7\ \&\ X>5)}{P(X>5)} = P(X > 7)/P(X > 5) = e^{-7/3}/e^{-5/3} = e^{-2/3} = .5134$.

Alternatively, we could use the memoryless property of exponential random variables, and compute $P(X > 7 \mid X > 5) = P(X > 2) = e^{-2/3} = .5134$.

2a. We have $P(X < Y) = \int_0^\infty \int_x^\infty 21e^{-3x-7y}\,dy\,dx = \int_0^\infty -3e^{-3x-7y}|_{y=x}^\infty\,dx = \int_0^\infty 3e^{-10x}\,dx = -3e^{-10x}/10|_{x=0}^\infty = 3/10$.
2b. We have $P(X < Y) = \int_0^\infty \int_x^\infty \lambda_1\lambda_2 e^{-\lambda_1 x-\lambda_2 y}\,dy\,dx = \int_0^\infty -\lambda_1 e^{-\lambda_1 x-\lambda_2 y}|_{y=x}^\infty\,dx = \int_0^\infty \lambda_1 e^{-(\lambda_1+\lambda_2)x}\,dx = -\lambda_1 e^{-(\lambda_1+\lambda_2)x}/(\lambda_1+\lambda_2)|_{x=0}^\infty = \lambda_1/(\lambda_1+\lambda_2)$.

3. We compute $P(Y < X < 2Y) = \int_0^\infty \int_y^{2y} 21e^{-3x-7y}\,dx\,dy = \int_0^\infty -7e^{-3x-7y}|_{x=y}^{2y}\,dy = \int_0^\infty (-7e^{-13y} + 7e^{-10y})\,dy = (7e^{-13y}/13 - 7e^{-10y}/10)|_{y=0}^\infty = (-7/13 + 7/10) = 21/130 = .1615$.

4a. Let $X_1, \ldots, X_5$ denote the respective times, in seconds, until children 1 through 5 take their next breaths. The desired probability is $P(X_1 > 1.2)P(X_2 > 1.2)\cdots P(X_5 > 1.2) = (e^{-1.2/3})^5 = .1353$.

Alternatively, the minimum of five independent exponential random variables is also an exponential random variable, with λ equal to the sum of the λ's from the five random variables; in this case, $\lambda = 1/3 + \cdots + 1/3 = 5/3$. So the desired probability is $e^{-(1.2)(5/3)} = .1353$.
4b. As in the "alternative" method for #4a, because the time until the next breath is an exponential random variable with parameter $\lambda = 1/3 + \cdots + 1/3 = 5/3$, then the expected time until the next child takes a breath is $1/\lambda = 3/5$ seconds.