STAT/MA 41600
In-Class Problem Set #39: November 13, 2017
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Problem Set 39 Answers

1a. We have $\text{Cov}(X_i, X_i) = \mathbb{E}(X_iX_i) - \mathbb{E}(X_i)\mathbb{E}(X_i) = \mathbb{E}(X_i) - \mathbb{E}(X_i)\mathbb{E}(X_i) = 4/52 - (4/52)^2 = 12/169$.
1b. We have $\text{Cov}(X_i, X_j) = \mathbb{E}(X_iX_j) - \mathbb{E}(X_i)\mathbb{E}(X_j) = (4/52)(3/51) - (4/52)^2 = -4/2873$.
1c. We have $\text{Var}(X) = \text{Cov}(X, X) = \text{Cov}(X_1 + \cdots + X_5, X_1 + \cdots + X_5) = 5\,\text{Cov}(X_1, X_1) + 20\,\text{Cov}(X_1, X_2) = 5(12/169) + 20(-4/2873) = 940/2873$.
2a. We have $\text{Cov}(X_i, X_i) = \mathbb{E}(X_iX_i) - \mathbb{E}(X_i)\mathbb{E}(X_i) = \mathbb{E}(X_i) - \mathbb{E}(X_i)\mathbb{E}(X_i) = 3/6 - (3/6)^2 = 1/4$.
2b. We have $\text{Cov}(X_i, X_j) = \mathbb{E}(X_iX_j) - \mathbb{E}(X_i)\mathbb{E}(X_j) = (3/6)(2/5) - (3/6)^2 = -1/20$.
2c. We have $\text{Var}(X) = \text{Cov}(X, X) = \text{Cov}(X_1 + \cdots + X_3, X_1 + \cdots + X_3) = 3\,\text{Cov}(X_1, X_1) + 6\,\text{Cov}(X_1, X_2) = 3(1/4) + 6(-1/20) = 9/20$.

3. We have $\text{Cov}(X, Y) = \int_0^2\int_{2y-4}^{8-4y}(x - 4/3)(y - 2/3)(1/12)\,dx\,dy = \int_0^2(x^2/2 - 4x/3)(y - 2/3)(1/12)|_{x=2y-4}^{8-4y}\,dy = \int_0^2(6y^2 - 16y + 8)(y - 2/3)(1/12)\,dy = \int_0^2(6y^3 - 20y^2 + 56y/3 - 16/3)(1/12)\,dy = (3y^4/2 - 20y^3/3 + 28y^2/3 - 16y/3)|_{y=0}^2(1/12) = -2/9$.

4. We have $\mathbb{E}(X) = \int_0^2(x)(3x^2/8)\,dx = 3x^4/32|_{x=0}^2 = 3/2$ and $\mathbb{E}(Y) = \int_0^2(y)(3/4)(2 - 2y + y^2/2)\,dy = (3/4)(y^2 - 2y^3/3 + y^4/8)|_{y=0}^2 = 1/2$.

So we conclude $\text{Cov}(X, Y) = \int_0^2\int_0^x(x - 3/2)(y - 1/2)(3/4)(x - y)\,dy\,dx = \int_0^2(x - 3/2)(3/4)\int_0^x(xy - x/2 - y^2 + y/2)\,dy\,dx = \int_0^2(x - 3/2)(3/4)(xy^2/2 - xy/2 - y^3/3 + y^2/4)|_{y=0}^x\,dx = \int_0^2(x - 3/2)(3/4)(x^3/6 - x^2/4)\,dx = \int_0^2(3/4)(x^4/6 - x^3/2 + 3x^2/8)\,dx = (3/4)(x^5/30 - x^4/8 + x^3/8)|_{x=0}^2 = 1/20$.