STAT/MA 41600
In-Class Problem Set #43: December 4, 2017
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Problem Set 43 Answers

1a. The MGF $M_X(t)$ of X is $\int_0^\infty (e^{tx})(\lambda e^{-\lambda x})\,dx = \int_0^\infty \lambda e^{(t-\lambda)x}\,dx = \frac{\lambda}{t-\lambda}e^{(t-\lambda)x}\big|_{x=0}^\infty = \lambda/(\lambda-t)$, which is valid for $t < \lambda$.
1b. We have $\mathbb{E}(X) = M_X'(0) = \frac{d}{dt}\frac{\lambda}{\lambda-t}\big|_{t=0} = \frac{\lambda}{(\lambda-t)^2}\Big|_{t=0} = 1/\lambda$.
1c. We have $\mathbb{E}(X^2) = M_X''(0) = \frac{d^2}{dt^2}\frac{\lambda}{\lambda-t}\Big|_{t=0} = \frac{d}{dt}\frac{\lambda}{(\lambda-t)^2}\Big|_{t=0} = \frac{2\lambda}{(\lambda-t)^3}\Big|_{t=0} = 2/\lambda^2$.

2a. We have $\mathbb{E}(X^3) = M_X'''(0) = \frac{d^3}{dt^3}\frac{\lambda}{\lambda-t}\Big|_{t=0} = \frac{d^2}{dt^2}\frac{\lambda}{(\lambda-t)^2}\Big|_{t=0} = \frac{d}{dt}\frac{2\lambda}{(\lambda-t)^3}\Big|_{t=0} = \frac{6\lambda}{(\lambda-t)^4}\Big|_{t=0} = 6/\lambda^3$.
2b. We have $\mathbb{E}(X^4) = M_X''''(0) = \frac{d^4}{dt^4}\frac{\lambda}{\lambda-t}\Big|_{t=0} = \frac{d^3}{dt^3}\frac{\lambda}{(\lambda-t)^2}\Big|_{t=0} = \frac{d^2}{dt^2}\frac{2\lambda}{(\lambda-t)^3}\Big|_{t=0} = \frac{d}{dt}\frac{6\lambda}{(\lambda-t)^4}\Big|_{t=0} = \frac{24\lambda}{(\lambda-t)^5}\Big|_{t=0} = 24/\lambda^4$.
2c. Yes, we continue the same line of reasoning, and we get $\mathbb{E}(X^n) = M_X^{(n)}(0) = \frac{n!\lambda}{(\lambda-t)^{n+1}}\Big|_{t=0} = n!/\lambda^n$.

3a. We have $\mathbb{E}(X) = \frac{d}{dt}(1-2t)^{-k/2}\big|_{t=0} = (-k/2)(1-2t)^{-k/2-1}(-2)\big|_{t=0} = k$.
3b. We have

$$\begin{aligned}\mathbb{E}(X^2) &= \frac{d^2}{dt^2}(1-2t)^{-k/2}\Big|_{t=0}\\ &= \frac{d}{dt}(k)(1-2t)^{-k/2-1}\Big|_{t=0}\\ &= (k)(-k/2-1)(1-2t)^{-k/2-2}(-2)\big|_{t=0}\\ &= (k)(k+2) = k^2+2k\end{aligned}$$

3c. We have $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = k^2 + 2k - (k)^2 = 2k$.

4a. All values $P(X = x)$ are nonnegative.
We also have $P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) = (125/156)(1/5)^0 + (125/156)(1/5)^1 + (125/156)(1/5)^2 + (125/156)(1/5)^3 = 125/156 + 25/156 + 5/156 + 1/156 = 1$.
For these two reasons, this is a valid probability mass function.
4b. The expected value of X is $\mathbb{E}(X) = (0)(125/156) + (1)(25/156) + (2)(5/156) + (3)(1/156) = 19/78$.
4c. The MGF $M_X(t)$ of X can be written as $M_X(t) = \sum_{x=0}^3 (e^{tx})(125/156)(1/5)^x$.
We can also go ahead and expand the summation, by writing: $M_X(t) = 125/156 + (e^t)(125/156)(1/5) + (e^{2t})(125/156)(1/5)^2 + (e^{3t})(125/156)(1/5)^3 = (125/156)(1 + (e^t/5) + (e^t/5)^2 + (e^t/5)^3)$.
Now we can multiply and divide by $1 - e^t/5$ to get $M_X(t) = (\frac{125}{156})(\frac{1-(e^t/5)^4}{1-e^t/5})$.
4d. We get $\mathbb{E}(X) = M_X'(0) = \frac{d}{dt}(\frac{125}{156})\frac{1-(e^t/5)^4}{1-e^t/5}\Big|_{t=0} = (\frac{125}{156})\frac{(1-e^t/5)(-4)(e^t/5)^3(e^t/5)-(1-(e^t/5)^4)(-e^t/5)}{(1-e^t/5)^2}\Big|_{t=0} = (\frac{125}{156})\frac{(1-1/5)(-4)(1/5)^3(1/5)-(1-(1/5)^4)(-1/5)}{(1-1/5)^2} = (\frac{125}{156})\frac{(4/5)(1/5)^3(-4/5)-(1-(1/5)^4)(-1/5)}{(4/5)^2} = 19/78$.