STAT/MA 41600
In-Class Problem Set #40: November 26, 2018
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Problem Set 40 Answers

1. The probability mass function $p_Y(y)$ of Y evaluated at $y = 1$ is
$p_Y(1) = \sum_{x=1}^{\infty}(11/16)(1/4)^{x-1}(1/3)^{1-1} = (11/16)/(1-1/4) = 11/12.$

So the conditional probability mass function of X, given $Y = 1$, is $p_{X|Y}(x \mid 1) = \frac{p_{X,Y}(x,1)}{p_Y(1)} = \frac{(11/16)(1/4)^{x-1}(1/3)^{1-1}}{11/12} = (3/4)(1/4)^{x-1}$ for $x \geq 1$, and $p_{X|Y}(x \mid 1) = 0$ otherwise.

Therefore, given $Y = 1$, the conditional distribution of X is Geometric with $p = 3/4$, so $\mathbb{E}(X \mid Y = 1) = 1/p = 4/3$.

2. The probability density function $f_Y(y)$ of Y evaluated at $y = 1$ is $f_Y(1) = \int_0^{25/3} 1/30\,dx$. (The upper limit on the integral comes from the fact that the hypotenuse of the triangle is the line $y = -(6/10)x + 6$, i.e., $x = 10 - (10/6)y$, so when $y = 1$ on this hypotenuse, we have $x = 10 - (10/6) = 50/6 = 25/3$.)

Therefore, we get $f_Y(1) = (25/3)(1/30) = 25/90 = 5/18$. So the conditional probability density function of X, given $Y = 1$, is $f_{X|Y}(x \mid 1) = \frac{f_{X,Y}(x,1)}{f_Y(1)} = \frac{1/30}{5/18} = 3/25$ for $0 \leq x \leq 25/3$, and $f_{X|Y}(x \mid 1) = 0$ otherwise.

Therefore, given $Y = 1$, the conditional distribution of X is Continuous Uniform on $[0, 25/3]$, so $\mathbb{E}(X \mid Y = 1) = (0 + 25/3)/2 = 25/6$.

3. The probability density function $f_X(x)$ of X evaluated at $x = 20$ is:
$f_X(20) = \int_{20}^{\infty}(1/750)e^{-(20/150+y/30)}\,dy = -(1/25)e^{-(2/15+y/30)}\big|_{y=20}^{\infty} = (1/25)e^{-(2/15+20/30)} = (1/25)e^{-4/5}.$

So the conditional probability density function of Y, given $X = 20$, is $f_{Y|X}(y \mid 20) = \frac{f_{X,Y}(20,y)}{f_X(20)} = \frac{(1/750)e^{-(20/150+y/30)}}{(1/25)e^{-4/5}} = (1/30)e^{-(y-20)/30}$ for $y > 20$, and $f_{Y|X}(y \mid 20) = 0$ otherwise.

Therefore, given $X = 20$, the conditional expected value of Y is
$\mathbb{E}(Y \mid X = 20) = \int_{20}^{\infty}(y)(1/30)e^{-(y-20)/30}\,dy = \int_0^{\infty}(y+20)(1/30)e^{-y/30}\,dy.$

We know that $\int_0^{\infty}(y)(1/30)e^{-y/30}\,dy = 30$ (this is the expected value of an Exponential random variable with $\lambda = 1/30$), and we know that $\int_0^{\infty}(20)(1/30)e^{-y/30}\,dy = 20\int_0^{\infty}(1/30)e^{-y/30}\,dy = 20$ (this is just integrating the probability density function of an Exponential random variable with $\lambda = 1/30$, and then multiplying by 20). So we conclude that $\mathbb{E}(Y \mid X = 20) = 30 + 20 = 50$.

Alternatively, we could have used integration by parts.

4. Let X denote the value on the blue die, and let Y denote the value of *the sum of the two dice.*

The probability mass function $p_Y(y)$ of Y evaluated at $y = 9$ is $p_Y(9) = 4/36$, since exactly 4 of the 36 equally likely results for the pair of dice will make the sum Y be equal to 9.

So the conditional probability mass function of X, given $Y = 9$, is $p_{X|Y}(x \mid 9) = \frac{p_{X,Y}(x,9)}{p_Y(9)} = \frac{1/36}{4/36} = 1/4$ for $x = 3, 4, 5, 6$, and $p_{X|Y}(x \mid 9) = 0$ otherwise. Therefore, given $Y = 9$, we compute $\mathbb{E}(X \mid Y = 9) = (1/4)(3) + (1/4)(4) + (1/4)(5) + (1/4)(6) = 9/2$.